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On Abstract Homomorphisms of Chevalley Groups with Nonreductive Image, I

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The efforts in the study of abstract homomorphisms between the groups of rational points of algebraic groups are aimed at proving that under certain conditions any group homomorphism μ: $G(k) \to G'(k')$, where G and G' are algebraic groups over (infinite) fields k and k', respectively, can be obtained from a field homomorphism $k \to k'$ and a k'-rational homomorphism ${}_{k'}G \to G'$, where ${}_{k'}G$ is obtained by the change of scalars from k to k' (such homomorphisms are called *standard*). In their fundamental paper [BoT], Borel and Tits showed, in particular, that if G and G' are absolutely simple, G is k-isotropic, and μ has a Zariski dense image, then any homomorphism μ is (basically) standard [BoT, 8.1]. In fact, the main result of [BoT] is more general and describes abstract homomorphisms when only G is assumed to be absolutely simple (and k-isotopic) while G' is allowed to be an arbitrary reductive group, but its statement is more technical (cf. [BoT, 8.16]). In the same paper ([BoT, 8.18]) Borel and Tits pointed out that dropping the assumption that G' is reductive opens a way to the existence of essentially new homomorphisms. Namely, given a field extension K/k and a derivation δ: $k \to K$, for any algebraic group G

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defined over the field of constants $k_0 = \{x \in k \mid \delta(x) = 0\}$, one can consider a homomorphism

$$\eta_\delta\colon G(k) \to G(K) \ltimes \mathfrak{g}(K), \qquad \eta_\delta(g) = (g, \Omega_\delta(g)),$$

where the semi-direct product is formed using the action of G on its Lie algebra $\mathfrak{g}$ via the adjoint representation, $\Omega_\delta(g) = g^{-1}\Phi_\delta(g)$, and $\Phi_\delta(g)$ is obtained by applying δ to every matrix entry of g; moreover, if δ is nontrivial, η_δ has a Zariski dense, hence nonreductive (as $\mathfrak{g}$ is the unipotent radical of $G \ltimes \mathfrak{g}$), image (for details cf. Section 2). In [T], Tits formulated a general conjecture that, under sufficiently general hypotheses on G and k and without assuming G' to be reductive, for any abstract homomorphism $\varphi\colon G(k) \to G'(k')$ there should exist a commutative finite-dimensional k'-algebra A and a ring homomorphism $\alpha\colon k \to A$ such that φ can be written as $\varphi = \psi \circ r_{A/k'} \circ \hat{\alpha}$, where $\hat{\alpha}\colon G(k) \to {}_AG(A)$ is induced by α (${}_AG$ is the group obtained by the change of scalars), $r_{A/k'}\colon {}_AG(A) \to R_{A/k'}({}_AG)(k')$ is the canonical isomorphism ($R_{A/k}$ is the functor of restriction of scalars), and ψ is a rational k'-morphism of $R_{A/k'}({}_AG)$ to G'. In the same paper Tits proved this conjecture for $k = k' = \mathbb{R}$ and also announced its truth for G a simple simply connected split k-group if k is not a nonperfect field of characteristic two. However, this result still leaves open the question about an explicit description of abstract homomorphisms as one would like to know precisely which algebras A and which rational homomorphisms ψ can actually arise. Subsequently, abstract homomorphisms with nonreductive images were not analyzed (to the best of our knowledge).

The goal of this paper is twofold. First, we show that if G is an absolutely simple simply connected split (in other words, Chevalley) group over a field k of characteristic zero, then any homomorphism of $G(k)$ such that the Zariski closure of its image has a commutative unipotent radical can be obtained from Borel–Tits' construction. This result does not depend on Tits' result [T] and gives an explicit description of such homomorphisms. Second, we describe a generalization of Borle–Tits' construction which allows one to construct abstract homomorphisms for which the unipotent radical of the Zariski closure of the image has arbitrarily large dimension and nilpotency class.

For abstract homomorphisms whose image has a commutative unipotent radical we prove the following.

Theorem 3. *Let G be a simple simply connected Chevalley group over a field k of characteristic zero. Furthermore, let $\mathcal{G}$ be a connected algebraic group over an extension K of k and $\mu\colon G(k) \to \mathcal{G}(K)$ be an abstract homomorphism* *with Zariski dense in $\mathcal{G}$ image. Assume that:*

(1) *the unipotent radical $V = R_u(\mathcal{G})$ is commutative, and*

(2) *if $G' = \mathcal{G}/V$, then the composition $G(k) \to \mathcal{G}(K) \to G'(K)$ of μ with the canonical morphism $\mathcal{G} \to G'$ extends to a rational K-homomorphism $\lambda: G \to G'$.*

Then

(i) *there exists a finite extension L/K over which $V = V_1 \oplus \cdots \oplus V_r$, where all V_i's are copies of the adjoint representation of G';*

(ii) *let $\mathcal{H} = G \ltimes (\underbrace{\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}}_{r})$, where $\mathfrak{g}$ is the Lie algebra of G on which G acts via the adjoint representation; then there exist derivations $\delta_1, \ldots, \delta_r: k \to L$ and an isogeny $\tau: \mathcal{H} \to \mathcal{G}$ such that $\mu = \tau \circ \eta_{\delta_1,\ldots,\delta_r}$ where $\eta_{\delta_1,\ldots,\delta_r}: G(k) \to \mathcal{H}(L)$ is defined by*

$$\eta_{\delta_1,\ldots,\delta_r}(g) = (g, \Omega_{\delta_1}(g), \ldots, \Omega_{\delta_r}(g)).$$

We fix notations in Section 1, and after some preparations in Sections 2 and 3 we prove Theorem 3 in Section 4. The latter section also contains some applications of Theorem 3, among which we mention here the following.

THEOREM 4 ("Superrigidity"). *Let G be a simple simply connected Chevalley group over a finitely generated field k of characteristic zero, $d = \mathrm{tr.deg.}_{\mathbb{Q}} k$. There exists an algebraic k-group $\mathcal{G}_0$ of dimension $(d+1) \cdot \dim G$ having a commutative unipotent radical and a group homomorphism $\iota: G(k) \to \mathcal{G}_0(k)$ with Zariski dense in $\mathcal{G}_0$ image such that, given an abstract homomorphism $\mu: G(k) \to \mathcal{G}(K)$ as in Theorem 1, there exists a unique rational K-homomorphism $\rho: \mathcal{G}_0 \to \mathcal{G}$ such that $\mu = \rho \circ \iota$.*

In Section 5 we give a sufficient condition for the unipotent radical of the image of an abstract homomorphism to be commutative. Finally, in Section 6 we generalize Borel–Tits' construction as described in the following theorem.

THEOREM 6. *Suppose there exists a nonzero derivation $\delta: k \to K$, and let k_0 denote the field of constants of δ. Given a connected algebraic k_0-group G, for any $n \geq 1$ one can construct a connected k_0-group $\mathcal{G}_n$ of dimension $(n+1) \cdot \dim G$ such that there exists an abstract homomorphism $G(k) \to \mathcal{G}_n(K)$ with a Zariski dense image. If G is reductive, then the unipotent radical $R_u(\mathcal{G}_n)$ has dimension $n \cdot \dim G$; if moreover G is semi-simple, then the nilpotency class of $R_u(\mathcal{G}_n)$ is n.*

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1. NOTATIONS AND CONVENTIONS

Everywhere in the paper G will denote a simple simply connected Chevalley group scheme. We let T denote a (fixed) split torus in G and let $R = R(T, G)$ be the corresponding root system. We fix an ordering on R, and denote by Π (resp., R_+) the set of simple (resp., positive) roots relative to this ordering. We will denote the Lie algebra of G by $\mathfrak{g}$, and let $\{X_\alpha\}_{\alpha\in R} \cup \{H_\alpha\}_{\alpha\in\Pi}$ denote a Chevalley basis of $\mathfrak{g}$ corresponding to the choice of the maximal torus and the ordering on R. For $\alpha \in R$, we let

$$u_\alpha(t) = \exp(tX_\alpha)$$

denote the canonical parametrization of the corresponding one-parameter unipotent subgroup $U_\alpha \subset G$. Furthermore, G_α will denote the subgroup (isomorphic to SL_2) of G generated by U_α and $U_{-\alpha}$, $T_\alpha = T \cap G_\alpha$, and $\mathfrak{g}_\alpha$ will be the Lie algebra of G_α.

Throughout the paper the ground field k will always have characteristic zero. Whenever convenient, we will tacitly identify G with the group of its points over a suitable algebraically closed field containing k.

Given an abstract homomorphism $\mu: G(k) \to \mathscr{G}(K)$ into the group of rational points of an algebraic group $\mathscr{G}$ over a field extension K/k, we will typically assume that the image of μ is Zariski dense in $\mathscr{G}$ and will use the corresponding script letters to denote the Zariski closure of the image of objects associated with G, e.g., $\mathscr{T} = \overline{\mu(T(k))}$, $\mathscr{G}_\alpha = \overline{G_\alpha(k)}$, etc. Since $G(k)$ does not have proper normal subgroups of finite index, the assumption about the density of the image of μ implies that $\mathscr{G}$ is automatically connected. Furthermore, since $G(k)$ is its own commutator subgroup, the radical $R(\mathscr{G})$ coincides with the unipotent radical $V = R_u(\mathscr{G})$. *Everywhere in this paper we will assume that the homomorphism $G(k) \to G' = \mathscr{G}/V$ obtained by composing μ with the canonical projection $\mathscr{G} \to \mathscr{G}/V$ extends to a rational homomorphism $\lambda: G \to G'$.* Then, in particular, for any connected k-subgroup $H \subset G$, the subgroup $\overline{\mu(H(k))}$ is also connected (as is any unipotent group in characteristic zero). It follows from the Levi decomposition (cf. [Mo]) that G' can be identified with a subgroup of $\mathscr{G}$, and then $\mathscr{G} = G' \ltimes V$. Then μ admits a presentation of the form

$$\mu(g) = (\lambda(g), v(g)) \quad \text{for } g \in G(k), \tag{1}$$

with $v(g) \in V$, which will be used throughout the paper.

Let λ: $G \to G'$ be a k-isogeny. Then $'$ (prime) will generally be used to label objects pertaining to G' (e.g., $\mathfrak{g}'$ will denote the Lie algebra of G'). However, we will not distinguish between the root systems $R(T, G)$ and $R(T', G')$, where $T' = \lambda(T)$; more precisely, we will identify these using the homomorphism λ^*: $X(T') \to X(T)$ of the character groups induced by λ. The differential $d\lambda$: $\mathfrak{g} \to \mathfrak{g}'$ is an isomorphism of the Lie algebras, and $\{X'_\alpha\}_{\alpha \in R} \cup \{H'_\alpha\}_{\alpha \in \Pi}$, where $X'_\alpha = d\lambda(X_\alpha)$ and $H'_\alpha = d\lambda(H_\alpha)$, is a Chevalley basis in $\mathfrak{g}'$. We also let $u'_\alpha(t) = \exp(tX'_\alpha)$ and note that $\lambda(u_\alpha(t)) = u'_\alpha(t)$.

2. DERIVATIONS AND HOMOMORPHISMS

For an extension K of k we let $K[\varepsilon]$, where $\varepsilon^2 = 0$, be the algebra of dual numbers over K. It is well-known (cf., for example, [J, Prop. 8.15]) that an additive function δ: $k \to K$ is a derivation if and only if the map $\tilde{\delta}$: $k \to K[\varepsilon]$, given by $\tilde{\delta}(x) = x + \delta(x)\varepsilon$, is a ring homomorphism. Moreover, given a derivation δ, the homomorphism $\tilde{\delta}$ is in fact a homomorphism of k_0-algebras where $k_0 = \{x \in k | \delta(x) = 0\}$ is the subfield of constants in k. It follows that for any algebraic k_0-group G one can consider the group homomorphism φ_δ: $G(k) \to G(K[\varepsilon])$ induced by the ring homomorphism $\tilde{\delta}$. If we fix a matrix realization of G, we can write φ_δ as

$$\varphi_\delta(g) = g + \Phi_\delta(g)\varepsilon,$$

for any $g \in G(k)$. Then $1 + g^{-1}\Phi_\delta(g)\varepsilon \in G(K[\varepsilon])$, i.e., $\Omega_\delta(g) := g^{-1}\Phi_\delta(g) \in \mathfrak{g}(K)$, where $\mathfrak{g}$ is the Lie algebra of G (cf. [Bo, 3.20]). Then

$$\eta_\delta\colon g \longmapsto (g, \Omega_\delta(g))$$

defines a group homomorphism $G(k) \to G(K) \ltimes \mathfrak{g}(K)$. The construction of η_δ is due to Borel and Tits (cf. [BoT, 8.18]) where it was also stated (without proof) that if $\delta \neq 0$, then η_δ has Zariski dense image (we will generalize this fact in Section 6). The goal of this section is to show in particular that any (abstract) homomorphism μ: $G(k) \to G(K) \ltimes \mathfrak{g}(K)$ such that $\mu(g) = (g, *)$ for all $g \in G(k)$ is of the form η_δ for a suitable derivation δ: $k \to K$. To formulate this result in a bit more general setting, we need one additional notation. Given an isogeny λ: $G \to G'$, we let $\mathscr{G} = G \ltimes \mathfrak{g}$, $\mathscr{G}' = G' \ltimes \mathfrak{g}'$, and denote by Λ: $\mathscr{G} \to \mathscr{G}'$ the rational homomorphism given by $\Lambda(g, X) = (\lambda(g), d\lambda(X))$.

THEOREM 1. *Suppose G is a simple simply connected Chevalley group over k. Let μ: $G(k) \to \mathscr{G}'(K) = G'(K) \ltimes \mathfrak{g}'(K)$, where K is an extension of k, be an abstract homomorphism of the form* (1), *Section* 1, *for a K-isogeny*

$\lambda\colon G \to G'$. *Then*

(1) *there exists* $v \in \mathfrak{g}'(K)$ *such that for* $a = (1, v)$ *one has* $a\mu(G(\mathbb{Z}))a^{-1} \subset G'(K)$;

(2) *for any* $\mu\colon G(k) \to \mathcal{G}'(K)$ *as above satisfying* $\mu(G(\mathbb{Z})) \subset G'(K)$ *there exists a unique derivation* $\delta\colon k \to K$ *such that* $\mu = \Lambda \circ \eta_\delta$.

Proof. We let $\Gamma = G(\mathbb{Z})$ if G has rank >1 and $\Gamma = G(\mathbb{Z}[\frac{1}{2}])$ if G has rank one, and we will think of Γ as a subgroup of $G(k)$. Then it follows either from the Superrigidity Theorem (cf. [Mar, Chap. VII, 3.10]) or from the positive solution of the congruence subgroup problem for Γ (cf. [BMS, Mat, and S]) that $\overline{\mu(\Gamma)}$ is a semi-simple K-defined subgroup of $\mathcal{G}'$[1] and therefore is conjugate to a subgroup of G' by an element of $R_u(\mathcal{G}')(K)$ (cf. [Mo, Sect. 7] and [BoS, Prop. 5.1]), so Part 1 follows.

Now, suppose a homomorphism $\mu\colon G(k) \to \mathcal{G}'(K)$ of the form (1), Section 1, satisfies

$$\mu(G(\mathbb{Z})) \subset G'(K). \tag{1}$$

Let α (resp., U) denote the maximal root (resp., the maximal unipotent subgroup) corresponding to the fixed ordering on R. Then the subgroups $U_\alpha(k)$ and $U(\mathbb{Z})$ commute elementwise, implying that $\mu(U_\alpha(k))$ and $\mu(U(\mathbb{Z}))$ will commute. Since $\mu(U(\mathbb{Z})) \subset G'$, this fact means that if $(*, X) \in \mu(U_\alpha(k))$, then X is fixed by $\lambda(U(\mathbb{Z}))$, so in view of the Zariski density of $U(\mathbb{Z})$ in U we obtain that X belongs to the centralizer of U' in $\mathfrak{g}'$, i.e., to $\langle X'_\alpha \rangle$, the subspace spanned by X'_α. This implies the existence of a function $\delta\colon k \to K$ such that

$$\mu(u_\alpha(a)) = (u'_\alpha(a), \delta(a)X'_\alpha). \tag{2}$$

In particular, $\mathcal{U}_\alpha = \overline{\mu(U_\alpha(K))}$ is contained in $\mathcal{U}'_\alpha := U'_\alpha \times \langle X'_\alpha \rangle$. Similarly, one proves that $\mathcal{U}_{-\alpha} = \overline{\mu(U_{-\alpha}(k))}$ is contained in $\mathcal{U}'_\alpha := U'_{-\alpha} \times \langle X'_{-\alpha} \rangle$. It follows that $\mathcal{G}_\alpha = \overline{\mu(G_\alpha(k))}$ is contained in $\mathcal{G}'_\alpha := G'_\alpha \ltimes \mathfrak{g}'_\alpha$, where $\mathfrak{g}'_\alpha$ is the Lie algebra of G'_α.

To identify $\overline{\mu(T_\alpha(k))}$, we need some additional notations. We let $\mathcal{N}_{\pm\alpha} = \{g \in \mathcal{G}'_\alpha | g^{-1}\mathcal{U}'_\alpha g \subset \mathcal{U}'_\alpha\}$ and $\mathcal{N} = \mathcal{N}_\alpha \cap \mathcal{N}_{-\alpha}$.

LEMMA 1. $\mathcal{N} = T'_\alpha \times \langle H'_\alpha \rangle$.

Proof. The inclusion $\mathcal{N} \supset T'_\alpha \times \langle H'_\alpha \rangle$ is obvious. Suppose $(g, X) \in \mathcal{N}$. Then $g \in G'$ normalizes both U'_α and $U'_{-\alpha}$, and therefore $g \in T'_\alpha$, hence $(1, X) \in \mathcal{N}$. Suppose $X = aX_{-\alpha} + bH_\alpha + cX_\alpha$. We wish to prove that $a = c = 0$. To prove that $c = 0$, we observe that since $(1, aX_\alpha + bH_\alpha) \in \mathcal{N}_\alpha$, we

[1]This can also be proved by purely combinatorial computations; cf. [St2] and [R].

obtain that $(1, cX_{-\alpha}) \in \mathcal{N}_\alpha$. However, as the following computation shows, the latter is impossible unless $c = 0$:

$$(1, -X_{-\alpha})(u_\alpha(1), 0)(1, X_{-\alpha}) = (u_\alpha(1), H_\alpha + X_\alpha).$$

Similarly, one finds that $a = 0$, proving the claim.

It follows from the lemma that $\mu(T_\alpha(k)) \subset T'_\alpha(K) \times KH'_\alpha$; i.e., there exists a function τ: $k^* \to K$ such that

$$\mu(h_\alpha(t)) = (\lambda(h_\alpha(t)), \tau(t)H'_\alpha),$$

h_α being the canonical parametrization of T_α (cf. [St1]).

Next, we will establish that the function δ introduced above is a derivation. It is a consequence of (1) that $\delta(1) = 0$, so the required fact follows from the following.

PROPOSITION 1. *Suppose δ: $k \to K$ and τ: $k^* \to K$ are two functions such that the maps*

$$U_\alpha(k) \xrightarrow{\mu_1} \mathcal{G}(K), \qquad u_\alpha(a) \mapsto (\lambda(u_\alpha(a)), \delta(a)X_\alpha),$$

and

$$T_\alpha(k) \xrightarrow{\mu_2} \mathcal{G}(K), \qquad h_\alpha(t) \mapsto (\lambda(h_\alpha(t)), \tau(t)H_\alpha),$$

extend to a homomorphism μ: $B_\alpha(k) = T_\alpha(k)U_\alpha(k) \to \mathcal{G}(K)$. If $\delta(1) = 0$, then

(1) *δ is a derivation*;

(2) *$\tau(t) = \delta(t)/t$ for all $t \in k^*$.*

Proof. Since μ_1 and μ_2 are homomorphisms, we obtain that δ is additive and τ satisfies $\tau(t_1t_2) = \tau(t_1) + \tau(t_2)$, implying that $\sigma(t) := t\tau(t)$ satisfies

$$\sigma(t_1t_2) = \sigma(t_1)t_2 + t_1\sigma(t_2) \qquad \text{for all } t_1, t_2 \in k^*. \tag{3}$$

Next, applying μ to the identity $h_\alpha(t)u_\alpha(a)h_\alpha(t)^{-1} = u_\alpha(at^2)$, after computations in $\mathcal{G}(K)$ we obtain

$$\delta\big(at^2\big) = \delta(a)t^2 + 2at^2\tau(t). \tag{4}$$

For the first assertion we need to establish the "product rule" for δ,

$$\delta(ab) = \delta(a)b + a\delta(b) \tag{5}$$

for all $a, b \in k$. Letting $a = 1$ in (4) and taking into account (3), we obtain

$$\delta\big(t^2\big) = 2t^2\tau(t) = 2t\sigma(t) = \sigma\big(t^2\big) \tag{6}$$

for all $t \in k^*$. If at least one of a or b equals ± 1, (5) is immediate. Otherwise we will write

$$a = a_+^2 - a_-^2, \qquad b = b_+^2 - b_-^2,$$

where

$$a_\pm = \frac{a \pm 1}{2}, \qquad b_\pm = \frac{b \pm 1}{2},$$

observing that none of $a_\pm, b_\pm$ is zero. Then using the additivity of δ we obtain

$$\delta(ab) = \delta\big(a_+^2 b_+^2\big) + \delta\big(a_-^2 b_-^2\big) - \delta\big(a_+^2 b_-^2\big) - \delta\big(a_-^2 b_+^2\big). \tag{7}$$

On the other hand, using (6) and then (3), we obtain

$$\delta\big(a_+^2 b_+^2\big) = \sigma\big(a_+^2 b_+^2\big) = \sigma\big(a_+^2\big) b_+^2 + a_+^2 \sigma\big(b_+^2\big) = \delta\big(a_+^2\big) b_+^2 + a_+^2 \delta\big(b_+^2\big),$$

etc. Substituting these equations into (7), after obvious simplifications we obtain

$$\delta(ab) = \delta\big(a_+^2 - a_-^2\big)\big(b_+^2 - b_-^2\big) + \big(a_+^2 - a_-^2\big)\delta\big(b_+^2 - b_-^2\big) = \delta(a)b + a\delta(b),$$

proving (5), and therefore Assertion 1.

Since δ is a derivation, we have $\delta\big(t^2\big) = 2t\delta(t)$. On the other hand, in view of (5) and (3), $\delta\big(t^2\big) = \sigma\big(t^2\big) = 2t\sigma(t)$. Comparing these equations, we obtain $\delta(t) = \sigma(t)$ for all $t \in k^*$, implying Assertion 2.

LEMMA 2. $\eta_\delta(u_\alpha(t)) = (u_\alpha(t), \delta(t) X_\alpha)$.

Proof. We have

$$\begin{aligned} \varphi_\delta(u_\alpha(t)) &= \exp((t + \delta(t)\varepsilon) X_\alpha) = \exp(t X_\alpha) \exp(\delta(t)\varepsilon X_\alpha) \\ &= u_\alpha(t)(1 + \delta(t)\varepsilon X_\alpha), \end{aligned}$$

implying the required result.

Now, we observe that, since δ is trivial on $\mathbb{Z}$, for any $g \in G(\mathbb{Z})$ we have

$$(\Lambda \circ \eta_\delta)(g) = (\lambda(g), 0) = \mu(g). \tag{8}$$

To conclude that $\mu = \Lambda \circ \eta_\delta$, we need the following.

LEMMA 3. *For any long root $\alpha \in R(T, G)$ the subgroups $G(\mathbb{Z})$ and $U_\alpha(k)$ generate $G(k)$.*

Proof. We need the following two facts: (1) $G(\mathbb{Z})$ contains representatives of all cosets in the Weyl group (cf. [St1, Sect. 3]) and (2) the Weyl group acts transitively on roots of the same length (cf. [Bou, Chap. VI, Sect. 1, Prop. 10]). This implies that the subgroup $H \subset G(k)$ generated by $G(\mathbb{Z})$ and $U_\alpha(k)$ contains $U_\beta(k)$ for *all* long $\beta \in R(T, G)$. In particular, if all roots have the same length, then immediately $H = G(k)$ as $U_\beta(k)$ for all $\beta \in R(T, G)$ generates $G(k)$ (since G is simply connected). Next, if $R(T, G)$ contains roots having different length, we choose a short root β not orthogonal to α. We claim that

$$U_\beta(k) - \{1\} = \{t u_\beta(1) t^{-1} | t \in T_\alpha(k)\}. \tag{9}$$

This follows from the relation

$$h_\alpha(t) u_\beta(1) h_\alpha(t)^{-1} = u_\beta(t^{\langle \beta, \alpha \rangle})$$

(cf. [St1, Lemma 20(c)]) as $\langle \beta, \alpha \rangle = 1$ because α is a long root and β is a short one. Since $T_\alpha(k) \subset H$, we obtain from (9) that $U_\beta(k) \subset H$. Using transitivity of the action of the Weyl group on all short roots, we obtain that H contains $U_\gamma(k)$ for all $\gamma \in R(T, G)$, hence $H = G(k)$.

Remark. Using commutator relations in Chevalley groups (particularly those described in Lemma 33 (for B_2) and in statement (3) on p. 151 (for G_2) of [St1]), one can show that the assertion of our Lemma 3 remains true in characteristic zero also for a short root α.

Now, we are ready to prove that $\mu = \Lambda \circ \eta_\delta$. By Lemma 2 and (8), μ and $\Lambda \circ \eta_\delta$ coincide on $G(\mathbb{Z})$ and $U_\alpha(k)$, and therefore it follows from Lemma 3 that they coincide everywhere, completing the proof.

3. THE CASE OF A COMMUTATIVE UNIPOTENT RADICAL

Let μ: $G(k) \to \mathscr{G}(K) = G'(K) \ltimes V(K)$ be an abstract homomorphism of the form (1), Section 1, associated to a K-isogeny λ: $G \to G'$. *In this section, if not otherwise stated, the unipotent radical V of the image group $\mathscr{G}$ will be assumed to be commutative* (*and nontrivial*). Then V is a vector group which can be considered as a G'-module. The goal of this section is to determine for which V a homomorphism μ as above can have Zariski dense image.

PROPOSITION 2. *Let G be a simple simply connected split group. If $V \neq 0$ is an* (*absolutely*) *irreducible G'-module such that there exists an abstract homomorphism μ: $G(k) \to \mathscr{G}(k) = G'(K) \ltimes V(K)$ of the form* (1), *Section* 1, *with Zariski dense image, then V is the adjoint representation of G'.*

We begin with a couple of lemmas. For a root $\alpha \in R(T, G)$ we let W_α denote the kernel of the projection $\mathcal{U}_\alpha \to U'_\alpha = \lambda(U_\alpha)$ induced by the projection $\mathcal{G} \to G'$ (obviously, $W_\alpha \subset V$).

LEMMA 4. *Let G be an arbitrary simple simply connected split group as above. Suppose $V = V_1 \oplus \cdots \oplus V_r$, where V_i's are irreducible G'-modules, and that μ: $G(k) \to G' \ltimes V$ is an abstract homomorphism of the form* (1), *Section* 1. *Let $\tilde{\alpha}$* (*resp*., U) *denote the maximal root* (*resp*., *the maximal unipotent subgroup*) *corresponding to the ordering on R. Then*

(i) *$W_{\tilde{\alpha}}$ is contained in the fixed subspace $V^{U'}$ of $U' = \lambda(U)$*;

(ii) *for any long root $\alpha \in R(T, G)$, one has* $\dim \mathcal{U}_\alpha \leq r + 1$; *in particular, if V is irreducible, then* $\dim \mathcal{U}_\alpha \leq 2$.

Proof. The fact that $U_{\tilde{\alpha}}$ is contained in the center of U implies that $\mu(U_{\tilde{\alpha}}(k))$, hence also $W_{\tilde{\alpha}}$, is contained in the center of $\overline{\mu(U(k))}$. However, since $(1, w) \in W_{\tilde{\alpha}}$ commutes with $\overline{\mu(U(k)), w}$ is contained in the fixed subspace $V^{U'}$ of U' = projection of $\overline{\mu(U(k))}$ to G', proving (i). Since for an irreducible G'-module the U'-fixed subspace is one-dimensional (cf. [H, 31.3]), we obtain $\dim W \leq \dim V^{U'} = r$, and therefore $\dim \mathcal{U}_{\tilde{\alpha}} \leq r + 1$. Since U_α for a long root α is conjugate by an element of $G(k)$ to $U_{\tilde{\alpha}}$, this implies our claim.

Next, we will establish a bound on the dimension of $\mathcal{G}$ in terms of that of $\mathcal{U}_\alpha$, where α is a long root. It is important to emphasize that *the following lemma is true without the assumption that the unipotent radical V is commutative*.

LEMMA 5. *Let $\alpha \in R(T, G)$ be a long root. If* $\dim \overline{\mu(U_\alpha(k))} = d$, *then* $\dim \overline{\mu(G(k))} \leq d \cdot \dim G$. *In addition, if there exists a root* $\beta \in R(T, G)$ *for which the dimension of $\mathcal{T}_\beta$ or $\mathcal{U}_\beta$ is $<d$, then* $\dim \overline{\mu(G(k))} < d \cdot \dim G$.

Proof. This relies on one fact for $G = SL_2$ which again is true without any assumptions about the unipotent radical. To formulate it, we denote by T (resp. U) the group of diagonal (resp. upper unitriangular) matrices in SL_2.

LEMMA 6. *Let $G = SL_2$ and μ: $G(k) \to \mathcal{G}(K)$ be an abstract homomorphism of the form* (1), *Section* 1. *Then*

$$\dim \overline{\mu(T(k))} \leq \overline{\mu(U(k))}.$$

Proof. Let

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

First, we claim that for $\mathcal{T} = \overline{\mu(T(k))}$, its intersection with the centralizer $Z_{\mathcal{G}}(\mu(u))$ has order ≤ 2. For this we observe that conjugation by $\mu(w)$

inverts any element of $\mathcal{T}$ and takes $\mu(u)$ to $\mu(v)$. It follows that any element $t \in \mathcal{T}$ commuting with $\mu(u)$ has to commute also with $\mu(v)$. But $\mu(w) \in \langle \mu(u), \mu(v) \rangle$, so $t = \mu(w) t \mu(w)^{-1} = t^{-1}$, i.e., $t^2 = 1$. Since $\mathcal{T}$ is commutative, all its elements satisfying this relation form a subgroup consisting of semi-simple elements. This subgroup projects injectively to $\mathcal{G}/R_u(\mathcal{G})$, from which the required assertion easily follows. Now, consider the map

$$\phi \colon \mathcal{T} \to \mathcal{U} = \overline{\mu(U(k))}, \qquad t \mapsto [t, u].$$

It follows from what we have previously established that for any $t \in \mathcal{T}$, the fibre $\phi^{-1}(\phi(t))$ has no more than two elements. So, by the dimension theorem, $\dim \mathcal{T} \leq \dim \mathcal{U}$, as claimed.

Returning to the proof of Lemma 5, we conclude from Lemma 6 that $\dim \overline{\mu(T_\alpha(k))} \leq d$, and the same is true for any long root. Let $\beta \in R(T, G)$ be a short root. Pick a long root α not orthogonal to β; then it follows from (9), Section 2, that $\dim \overline{\mu(U_\beta(k))} \leq \dim \overline{\mu(T_\alpha(k))} \leq d$. Applying Lemma 6 one more time, we obtain $\dim \overline{\mu(T_\beta(k))} \leq d$. Since

$$U(k) = \prod_{\gamma > 0} U_\gamma(k) \qquad \text{and} \qquad T(k) = \prod_{\gamma \text{ simple}} T_\gamma(k),$$

we conclude that

$$\dim U \leq d \cdot (\text{\# of positive roots}) \qquad \text{and} \qquad \dim T \leq d \cdot (\text{rank of } G).$$

Now, using the Bruhat decomposition

$$G(k) = \bigcup_w U(k) T(k) w U(k)$$

(w runs through a set of representatives, taken from $G(k)$, of all elements of the Weyl group), we obtain

$$\begin{aligned} \dim \overline{\mu(G(k))} &\leq d \cdot (2 \cdot (\text{\# of positive roots}) + (\text{rank of } G)) \\ &= d \cdot \dim G. \end{aligned}$$

Finally, if for some root $\beta \in R(T, G)$ the dimension of $\mathcal{T}_\beta$ or $\mathcal{U}_\beta$ is $<d$, then the same is true for all roots of the same length as β; in particular, there is always a simple root with this property. Then the above method using the Bruhat decomposition gives a sharper bound: $\dim \overline{\mu(G(k))} < d \cdot \dim G$.

Proof of Proposition 2

Case $G = SL_2$. It follows from Lemma 4 that $\dim \overline{\mu(U(k))} \leq 2$, and then according to Lemma 5 we have $\dim \mathscr{G} \leq 6$; i.e., $\dim V \leq 3$. Obviously, the case $\dim V = 1$ cannot occur in our setup. Let us also eliminate the case $\dim V = 2$. In this case λ must be an isomorphism, so we may identify G' with G and assume that $\lambda = \mathrm{id}_G$, and V must be the standard representation of $G = SL_2$. Let $z = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and let $\mu(z) = (z, v)$. Since z belongs to the center of $G(k)$ and $\mu(G(k))$ is Zariski dense in $\mathscr{G}$, $\mu(z)$ must belong to the center of $\mathscr{G}$, in particular, must commute with V. It follows that z must act on V trivially, while in fact z acts as multiplication by -1, a contradiction. Thus, the only remaining possibility is $\dim V = 3$ in which case V corresponds to the adjoint representation of G (or G').

General Case. Again by Lemma 4, $d = \dim \overline{\mu(U_\alpha(k))}$ cannot exceed two. If $d = 1$, then by Lemma 5

$$\dim \overline{\mu(G(k))} \leq \dim G,$$

so $\mu(G(k))$ cannot be dense in $\mathscr{G}$ as $V \neq 0$. Thus, $d = 2$.

Now, we fix an ordering on $R(T, G)$ and let $\tilde{\alpha}$ (resp., U) denote the maximal root (resp., the maximal unipotent subgroup) corresponding to this ordering. Since $d = 2$, we conclude that $\dim W_{\tilde{\alpha}} = 1$ in the notations introduced prior to the statement of Lemma 4. On the other hand, according to Lemma 1(i), $W_{\tilde{\alpha}} \subset V^{U'}$, where $U' = \lambda(U)$. Since V is irreducible, $V^{U'}$ is one-dimensional and coincides with the eigenspace $V(\delta)$ of $T' = \lambda(T)$ corresponding to the highest weight δ of ρ. Thus,

$$W_{\tilde{\alpha}} = V(\delta). \tag{1}$$

We observe that $V_{\tilde{\alpha}} := R_u(\mathscr{G}_{\tilde{\alpha}})$ coincides with $\mathscr{G}_{\tilde{\alpha}} \cap V$ as $\mathscr{G}_{\tilde{\alpha}}/(\mathscr{G}_{\tilde{\alpha}} \cap V) \simeq G'_{\tilde{\alpha}}$ is reductive; in particular, $W_{\tilde{\alpha}} \subset V_{\tilde{\alpha}}$. Since $\dim U_{\tilde{\alpha}} = 2$, we conclude from the case of SL_2 that $V_{\tilde{\alpha}}$ is the adjoint representation of $G'_{\tilde{\alpha}} = \lambda(G_{\tilde{\alpha}})$, and moreover that $W_{\tilde{\alpha}}$ coincides with the eigenspace of $T'_{\tilde{\alpha}} = \lambda(T_{\tilde{\alpha}})$ of weight $\tilde{\alpha}$. Comparing with (1), we obtain that

$$\delta | T'_{\tilde{\alpha}} = \tilde{\alpha} | T'_{\tilde{\alpha}}.$$

Furthermore, let $S \subset T$ be the kernel of $\tilde{\alpha}$. Then $S(k)$ commutes with $G_{\tilde{\alpha}}(k)$, so $\mu(S(k))$ will commute with $\mathscr{G}_{\tilde{\alpha}}$, implying that $\lambda(S(k))$ acts on $W_{\tilde{\alpha}}$ trivially; i.e., $\delta(\lambda(S(k))) = 1$. We see that α and δ coincide on $T'_{\tilde{\alpha}}\lambda(S(k))$, which implies that $\delta = \tilde{\alpha}$ as $T'_{\tilde{\alpha}}\lambda(S(k))$ is Zariski dense in T'. This means that V is the adjoint representation of G' (or G).

THEOREM 2. *Let* μ: $G(k) \to \mathscr{G}(K) = G'(K) \ltimes V(K)$ *be an abstract homomorphism of the form* (1), *Section* 1, *associated with a rational* *homomorphism $\lambda\colon G \to G'$. If μ has Zariski dense image, then*:

(i) *over a suitable finite extension L/K there exists a decomposition $V = V_1 \oplus \cdots \oplus V_r$ where each V_i is a copy of the adjoint representation of G'*;

(ii) $\dim \overline{\mu(T_\beta(k))} = \dim \overline{\mu(U_\beta(k))} = r + 1$ *for any $\beta \in R(T, G)$.*

Proof. It follows from complete reducibility of representations of reductive groups in characteristic zero (cf. [H, 14.3]) that there exists a finite extension L/K over which $V = V_1 \oplus \cdots \oplus V_r$, where each V_i is an absolutely irreducible G'-module; let $\pi_i\colon V \to V_i$ be the corresponding projection. Then the composite map

$$\mu_i\colon G(k) \xrightarrow{\mu} G'(L) \ltimes V(L) \xrightarrow{(id,\, \pi_i)} G'(L) \ltimes V_i(L) \tag{2}$$

has Zariski dense image, so it follows from Proposition 2 that V_i is the adjoint representation of G'.

For (ii) we observe that it follows from Lemma 4 that for a long root α one has $\dim \mathcal{U}_\alpha \le r + 1$. If there were a root $\beta \in R(T, G)$ for which the dimension of $\mathcal{T}_\beta$ or $\mathcal{U}_\beta$ is $<(r + 1)$, then we would obtain from Lemma 5 that

$$\dim \overline{\mu(G(k))} < (r + 1) \dim G = \dim \mathcal{G},$$

so $\mu(G(k))$ could not be possibly be dense in $\mathcal{G}$, a contradiction proving (ii).

4. THEOREM 3 AND ITS CONSEQUENCES

THEOREM 3. *Let G be a simple simply connected Chevalley group over a field k of characteristic zero. Furthermore, let $\mathcal{G}$ be a connected algebraic group over an extension K of k and $\mu\colon G(k) \to \mathcal{G}(K)$ be an abstract homomorphism with Zariski dense in $\mathcal{G}$ image. Assume that*:

(1) *the unipotent radical $V = R_u(\mathcal{G})$ is commutative, and*

(2) *if $G' = \mathcal{G}/V$, then the composition $G(k) \to \mathcal{G}(K) \to G'(K)$ of μ with the canonical morphism $\mathcal{G} \to G'$ extends to a rational K-homomorphism $\lambda\colon G \to G'$.*

Then

(i) *there exists a finite extension L/K over which $V = V_1 \oplus \cdots \oplus V_r$, where all V_i's are copies of the adjoint representation of G'*;

(ii) *let $\mathcal{H} = G \ltimes (\underbrace{\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}}_{r})$, where $\mathfrak{g}$ is the Lie algebra of G on which G acts via the adjoint representation; then there exist derivations*

$\delta_1, \ldots, \delta_r\colon k \to L$ and an L-isogeny $\tau\colon \mathscr{H} \to \mathscr{G}$ such that $\mu = \tau \circ \eta_{\delta_1,\ldots,\delta_r}$ where $\eta_{\delta_1,\ldots,\delta_r}\colon G(k) \to \mathscr{H}(L)$ is defined by

$$\eta_{\delta_1,\ldots,\delta_r}(g) = (g, \Omega_{\delta_1}(g), \ldots, \Omega_{\delta_r}(g)).$$

Proof. According to Part i, which has already been established in Theorem 2, we can identify each V_i with $\mathfrak{g}'$. Consider the homomorphism $\mu_i\colon G(k) \to G'(L) \ltimes V_i(L)$ given by (2), Section 3. By Theorem 1 there exist an element $a_i = (1, v_i) \in G'(L) \ltimes V_i(L)$ and a derivation $\delta_i\colon k \to L$ such that $\mu_i(g) = a_i^{-1}(\Lambda \circ \eta_{\delta_i})(g)a_i$ for all $g \in G(k)$. Then $\tau\colon \mathscr{H} \to \mathscr{G}$ defined by

$$\tau(g; x_1, \ldots, x_r) = a^{-1}(\lambda(g); d\lambda(x_1), \ldots, d\lambda(x_r))a,$$

where $a = (1; v_1, \ldots, v_r)$, is as required.

Now, let us specialize our considerations for the topological situation. Suppose k is a topological field and K/k is a topological field extension. Let $\mu\colon G(k) \to \mathscr{G}(K) = G'(K) \ltimes V(K)$ be a *continuous* homomorphism as in (1), Section 1. Pick a finite extension L/K over which $V = V_1 \oplus \cdots \oplus V_r$, the direct sum of copies of the adjoint representation of G', and extend the topology of K to L (for example, by identifying L with $K^{[L:K]}$). Since the topology on $V(L)$ is the direct product topology with respect to an arbitrary basis, it follows from (2), Section 2, that the derivations $\delta_1, \ldots, \delta_r\colon k \to L$ arising in the description of μ given in Theorem 3 are continuous (for *any* extension of the topology from K to L).

A remark regarding how Theorem 3 fits in with the conjecture of Tits mentioned in Section 1 is in order. Obviously, for a single derivation $\delta\colon k \to K$ the homomorphism $\eta_\delta\colon G(k) \to G(K) \ltimes \mathfrak{g}(K)$ can be decomposed as follows: $\eta_\delta = \psi \circ R_{K[\varepsilon]/K} \circ \varphi_\delta$, where $\varphi_\delta\colon G(k) \to G(K[\varepsilon])$ (here $\varepsilon^2 = 0$) is induced by the ring homomorphism $k \to K[\varepsilon]$, $x \longmapsto x + \delta(x)\varepsilon$, $R_{K[\varepsilon]/K}$ is the restriction of scalars, and $\psi\colon R_{K[\varepsilon]/K}(G) \to G \ltimes \mathfrak{g}$ is the rational map given by

$$g + \varepsilon X \longmapsto (g, g^{-1}X). \tag{1}$$

In the general case, given $\mu\colon G(k) \to G(K) \ltimes V(K)$, the decomposition $V = V_1 \oplus \cdots \oplus V_r$ into the direct sum of copies of the adjoint representation of G may require passing from K to its finite Galois extension L, whose Galois group we will denote by $\mathfrak{G}$. Then one can pick some $i_1, \ldots, i_l \in \{1, \ldots, r\}$ such that $V = W_1 \oplus \cdots \oplus W_l$, where W_j is the sum (not necessarily direct) of the Galois conjugates $V_{i_j}^{\sigma}$, $\sigma \in \mathfrak{G}$. There exists a K-defined homomorphism of $G(K)$-modules $\pi_j\colon R_{L/K}(V_{i_j}) \to W_j$. Furthermore, pick derivations $\delta_1, \ldots, \delta_r\colon k \to L$ such that (after possible conjugation) $\mu = \eta_{\delta_1,\ldots,\delta_r}$. Then μ can be decomposed as

$$\mu = \pi \circ \psi \circ R_{A/K} \circ \varphi_{\delta_{i_1},\ldots,\delta_{i_l}}, \tag{2}$$

where $A = K \oplus L\varepsilon_1 \oplus \cdots \oplus L\varepsilon_l$ and $\varepsilon_i\varepsilon_j = 0$ for all i, j, $\varphi_{\delta_{i_1},\dots,\delta_{i_l}}$: $G(k) \to G(A)$ is induced by the ring homomorphism $k \to A$, $x \mapsto x + \delta_{i_1}(x)\varepsilon_1 + \cdots + \delta_{i_l}(x)\varepsilon_l$, $R_{A/K}$ is the restriction of scalars, ψ: $R_{A/K}(G) \to G \ltimes R_{L/K}(\mathfrak{g})^l$ is similar to (1), and π: $G \ltimes R_{L/K}(\mathfrak{g})^l \to G \ltimes V$ is given by

$$\pi(g, w_1, \dots w_l) = (g, \pi_1(w_1) + \cdots + \pi_l(w_l)).$$

We remark however that in some instances the decomposition (2) is less convenient to work with than the presentation $\mu = \tau \circ \eta_{\delta_1,\dots,\delta_r}$ in Theorem 3 (though the latter may not be defined over K) as π may have a kernel of positive dimension (cf., for example, the proof of Theorem 5 below).

For further reference, we will record here a computation of the dimension of the image of $\eta_{\delta_1,\dots,\delta_l}$.

LEMMA 7. *Let $\delta_1, \dots, \delta_l$: $k \to K$ be a collection of derivations. Then*

$$\dim \overline{\eta_{\delta_1,\dots,\delta_l}(G(k))} = (r+1) \cdot \dim G,$$

where r is the number of linearly independent (over K) derivations among $\delta_1, \dots, \delta_l$ (i.e., the dimension of the K-linear span of $\delta_1, \dots, \delta_l$.).

Proof. If derivations $\delta_{r+1}, \dots, \delta_l$ are linear combinations of $\delta_1, \dots, \delta_r$, then the morphism

$$G \ltimes \mathfrak{g}^l \to G \ltimes \mathfrak{g}^r,$$

given by the projection $\mathfrak{g}^l \to \mathfrak{g}^r$ to the first r coordinates, induces an isomorphism between $\overline{\eta_{\delta_1,\dots,\delta_l}(G(k))}$ and $\overline{\eta_{\delta_1,\dots,\delta_r}(G(k))}$. So, it remains to be proved that if $\delta_1, \dots, \delta_r$ are linearly independent, then

$$\dim \overline{\eta_{\delta_1,\dots,\delta_r}(G(k))} = (r+1) \cdot \dim G.$$

According to Theorem 2(ii), it suffices to show that for a root $\alpha \in R(T, G)$ the group $\mathcal{U}_\alpha = \overline{\eta_{\delta_1,\dots,\delta_r}(U_\alpha(k))}$ has dimension $(r+1)$. It follows from Lemma 2 that

$$\eta_{\delta_1,\dots,\delta_r}(u_\alpha(t)) = (u_\alpha(t), \delta_1(t)X_\alpha, \dots, \delta_r(t)X_\alpha).$$

Since $(U_\alpha, 0) = \overline{\eta_{\delta_1,\dots,\delta_r}(U_\alpha(\mathbb{Q}))} \subset \mathcal{U}_\alpha$, we conclude that $\dim \mathcal{U}_\alpha = 1 + m$, where m is the dimension of the closure of the image of the additive map κ: $k \to \mathbb{A}^r$, $\kappa(t) = (\delta_1(t), \dots, \delta_r(t))$. However, since char $k = 0$, closed additive subgroups of $\mathbb{A}^r$ coincide with subspaces, so the fact that $m < r$ would mean that there is a linear relation between $\delta_1, \dots, \delta_r$. Thus, $m = r$, proving the lemma.

Remark. Derivations $\delta_1, \dots, \delta_r \colon k \to K$ are linearly independent if and only if there are points $x_1, \dots, x_r \in k$ such that the vectors

$$(\delta_1(x_1), \dots, \delta_1(x_r)), \dots, (\delta_r(x_1), \dots, \delta_r(x_r)) \in K^r \tag{3}$$

are linearly independent. One implication here is obvious; the other is an elementary fact from linear algebra which we will recall for the sake of completeness. Pick the largest l such that there exit $x_1, \dots, x_l \in k$ for which the vectors

$$(\delta_1(x_1), \dots, \delta_r(x_1)), \dots, (\delta_r(x_l), \dots, \delta_r(x_l)) \in K^r$$

are linearly independent. We may suppose that det $(\delta_i(x_j))_{i,j=1,\dots,l} \neq 0$. If $l < r$, there are $a_1, \dots, a_l \in K$ such that

$$\delta_{l+1}(x_j) = a_1\delta_1(x_j) + \cdots + a_l\delta_l(x_j) \qquad \text{for all } j = 1, \dots, l.$$

Since $\delta_1, \dots, \delta_l, \delta_{l+1}$ are linearly independent, there exists $x_{l+1} \in k$ such that

$$\delta_{l+1}(x_{l+1}) \neq a_1\delta_1(x_{l+1}) + \cdots + a_l\delta_l(x_{l+1}),$$

and then the vectors

$$(\delta_1(x_1), \dots, \delta_r(x_1)), \dots, (\delta_r(x_{l+1}), \dots, \delta_r(x_{l+1}))$$

are linearly independent, contradicting the maximality of l. So, $l = r$, and the vectors (3) are linearly independent.

THEOREM 4 ("Superrigidity"). *Let G be a simple simply connected Chevalley group over a finitely generated field k of characteristic zero, $d = \mathrm{tr.deg.}_{\mathbb{Q}} k$. There exist an algebraic k-group $\mathscr{G}_0$ of dimension $(d+1) \cdot \dim G$ having commutative unipotent radical and a group homomorphism $\iota \colon G(k) \to \mathscr{G}_0(K)$ with Zariski dense in $\mathscr{G}_0$ image such that, given an abstract homomorphism $\mu \colon G(k) \to \mathscr{G}(K)$ as in Theorem* 1, *there exists a unique rational K-homomorphism $\rho \colon \mathscr{G}_0 \to \mathscr{G}$ such that $\mu = \rho \circ \iota$.*

Proof. The proof of Theorem 8.42 in [J] implies that the space of derivations $\mathrm{Der}(k, K)$ has dimension d over K. We pick a basis $\theta_1, \dots, \theta_d$ of $\mathrm{Der}(k, K)$ and consider the homomorphism

$$\iota = \eta_{\theta_1,\dots,\theta_d} \colon G(k) \to \mathscr{G}_0(K),$$

where $\mathscr{G}_0 = G \ltimes \mathfrak{g}^d$. It follows from Lemma 7 that Im ι is Zariski dense in $\mathscr{G}_0$. Now, let $\mu \colon G(k) \to \mathscr{G}(K)$, where $\mathscr{G} = G' \ltimes V$, be an abstract homomorphism of the form (1), Section 1. We will assume (as we may) that μ has a Zariski dense image. Pick a factorization $\mu = \tau \circ \eta_{\delta_1,\dots,\delta_r}$ provided by

Theorem 3 where δ_i: $k \to L$ are appropriate derivations. Since $\theta_1, \dots, \theta_d$ form a basis of $\mathrm{Der}(k, L)$ over L as well, there are expressions

$$\delta_i = \sum_{j=1}^{d} \alpha_{ij}\theta_j$$

with $\alpha_{ij} \in L$. Let π: $\mathcal{G}_0 = G \ltimes \mathfrak{g}^d \to G \ltimes \mathfrak{g}^r$ be the morphism given by the equation

$$\pi(x; a_1, \dots, a_d) = \left(x; \sum_{j=1}^{d} \alpha_{1j}a_j, \dots, \sum_{j=1}^{d} \alpha_{rj}a_j\right).$$

Then $\mu = \rho \circ \iota$ for $\rho = \tau \circ \pi$. The uniqueness of ρ follows from the Zariski density of $\mathrm{Im}\ \iota$ in $\mathcal{G}_0$. By construction, ρ is defined at least over L. Moreover, $\iota(G(k)) \subset \mathcal{G}_0(K)$ and $\rho(\iota(G(k))) = \mu(G(k)) \subset \mathcal{G}(K)$. It follows that for any $\sigma \in \mathrm{Gal}(\overline{K}/K)$, ρ and its Galois conjugate ${}^{\sigma}\rho$ coincide on $\iota(G(k))$. By the uniqueness of ρ, we obtain ${}^{\sigma}\tau = \tau$, hence τ is K-defined (cf. [B, Sect. AG, 14.3]).

5. DETECTING HOMOMORPHISMS WHOSE IMAGE HAS COMMUTATIVE UNIPOTENT RADICAL

THEOREM 5. *Let μ: $G(k) \to \mathcal{G}(K)$ be an abstract homomorphism with Zariski dense image. If for a long root α, the unipotent radical of the group $\mathcal{G}_\alpha := \overline{\mu(G_\alpha(k))}$ is commutative, then the unipotent radical of $\mathcal{G}$ is also commutative.*

Proof. Assume that $V = R_u(\mathcal{G})$ is not commutative. Then replacing μ by its composition with the canonical rational homomorphism $\mathcal{G} \to \mathcal{G}/[V, [V, V]]$, we may suppose from the very beginning that $V' = [V, V]$ is central in V, in particular, is abelian. Let μ': $G(k) \to \mathcal{G}' := \mathcal{G}/V'$ denote the composition of μ with the canonical homomorphism $\mathcal{G} \to \mathcal{G}/V'$. We know from Theorem 3 that $V/V' = R_u(\mathcal{G}')$ can be written as

$$V/V' = V_1 \oplus \cdots \oplus V_r, \tag{1}$$

where V_i's are copies of the adjoint representation of G'.

Next, let $\mathcal{G}_\alpha = G'_\alpha \ltimes W$ be the Levi decomposition of $\mathcal{G}_\alpha$, where by our assumption W is commutative. Then

$$W = W_1 \oplus \cdots \oplus W_m,$$

where W_i's are copies of the adjoint representation of G'_α. It follows from Theorem 2 that

$$\dim \overline{\mu'(U_\alpha(k))} = r + 1 \qquad \text{and} \qquad \dim \overline{\mu(U_\alpha(k))} = m + 1,$$

implying, in particular, that $m \geq r$.

We have

$$\overline{\mu'(G_\alpha(k))} = G'_\alpha \ltimes (Z_1 \oplus \cdots \oplus Z_r),$$

where Z_i's are copies of the adjoint representation of G'_α (we observe that since $\dim \overline{\mu'(U_\alpha(k))} = r+1$, the number of Z_i's is equal to r, the same number as in (1). According to Theorem 3, there are presentations $\mu|_{G_\alpha(k)} = \sigma \circ \eta_{\omega_1,\ldots,\omega_m}$ and $\mu'|_{G_\alpha(k)} = \tau \circ \eta_{\zeta_1,\ldots,\zeta_r}$ for some isogenies $\sigma\colon G_\alpha \ltimes \mathfrak{g}_\alpha^m \to \mathscr{G}_\alpha$ and $\tau\colon G_\alpha \ltimes \mathfrak{g}_\alpha^r \to \overline{\mu'(G_\alpha(k))}$ and for some derivations $\omega_i\colon k \to L$ $(i = 1,\ldots,m)$ and $\zeta_i\colon k \to L$ $(i = 1,\ldots,r)$, where L is some finite extension of K. By Lemma 7, the derivations $\zeta_1,\ldots,\zeta_r$ are linearly independent, so by the remark following the lemma there are points $x_1,\ldots,x_r \in k$ such that the vectors

$$(\zeta_1(x_1),\ldots,\zeta_r(x_1)),\ldots,(\zeta_1(x_r),\ldots,\zeta_r(x_r)) \in K^r$$

are linearly independent. Then we pick $(m-r)$ linearly independent vectors

$$(b_{11},\ldots,b_{1m}),\ldots,(b_{m-r\,1},\ldots,b_{m-r\,m}) \in K^m$$

such that

$$\sum_{j=1}^m b_{ij}\omega_j(x_k) = 0 \qquad \text{for all } i = 1,\ldots,m-r \text{ and } k = 1,\ldots,r. \tag{2}$$

Consider the derivations $\delta_i = \sum_{j=1}^m b_{ij}\omega_j$, and let $k_0 \subset k$ denote the subfield consisting of elements on which all $\delta_1,\ldots,\delta_{m-r}$ vanish (if $r = m$, then by convention $k_0 = k$). It follows from our construction that $x_1,\ldots,x_r \in k_0$, implying that the restriction of $\zeta_1,\ldots,\zeta_r$ to k_0 remains linearly independent, and therefore $\dim \overline{\mu'(U_\alpha(k_0))} = r+1$. Now, by Theorem 2, the unipotent radical of $\overline{\mu'(G(k_0))}$ contains r copies of the adjoint representation of G', implying that $\overline{\mu'(G(k_0))} = \mathscr{G}/V'$. It follows that for $V_0 = \overline{\mu(G(k_0))} \cap V$ one has $V_0V' = V$. On the other hand, in view of the equations (2) among the restrictions of $\omega_1,\ldots,\omega_m$ to k_0 there are no more than r which are linearly independent, hence $\dim \overline{\mu(U_\alpha(k_0))} \leq r+1$. In view of Lemma 5 this implies that $\dim \overline{\mu(G(k_0))} \leq (r+1)\cdot \dim G$, i.e., $\dim V_0 \leq r \cdot \dim G = \dim V/V'$. It follows that $V = V_0 \times V'$. Then $V_0 \simeq V/V'$ is abelian, and therefore so is V, a contradiction.

6. ON CONSTRUCTING ABSTRACT HOMOMORPHISMS WITH NONREDUCTIVE IMAGE

The goal of this section is to generalize the construction of abstract homomorphisms with nonreductive image given by Borel and Tits (cf. [BoT, 8.18]).

THEOREM 6. *Suppose there exists a nonzero derivation* $\delta\colon k \to K$, *and let* k_0 *denote the field of constants of* δ. *Given a connected algebraic* k_0*-group* G, *for any* $n \geq 1$ *one can construct a connected* k_0*-group* $\mathcal{G}_n$ *of dimension* $(n+1)\cdot \dim G$ *such that there exists an abstract homomorphism* $G(k) \to \mathcal{G}_n(K)$ *with Zariski dense image. If* G *is reductive, then the unipotent radical* $R_u(\mathcal{G}_n)$ *has dimension* $n \cdot \dim G$; *if moreover* G *is semi-simple, then the nilpotency class of* $R_u(\mathcal{G}_n)$ *is* n.

The first step in the proof is a construction of certain ring homomorphisms attached to a derivation. Fix an $n \geq 1$ and consider the algebra $K[\varepsilon_n]$ where $\varepsilon_n^{n+1} = 0$. A given derivation $\delta\colon k \to K$ can be extended to a derivation $K \to K$ (cf. [J, Prop. 8.17]); though this extension is not necessarily unique, we will denote it also by δ (our construction goes through for *any* extension). We let K_0 denote the field of constants in K so that $k_0 = K_0 \cap k$. Then one defines $t_{\delta,n}\colon k \to K[\varepsilon_n]$ by

$$t_{\delta,n}(x) = x + \delta(x)\varepsilon_n + \frac{\delta^2(x)}{2!}\varepsilon_n^2 + \cdots + \frac{\delta^k(x)}{k!}\varepsilon_n^k + \cdots + \frac{\delta^n(x)}{n!}\varepsilon_n^n \quad (1)$$

and $\theta_{\delta,n}\colon K[\varepsilon_n] \to K[\varepsilon_n]$ by

$$\theta_{\delta,n}\left(\sum_{i=0}^{n} a_i\varepsilon_n^i\right) = \sum_{i=0}^{n}\left(\sum_{j=0}^{i}\frac{1}{j!}\delta^j(a_{i-j})\right)\varepsilon_n^i, \quad (2)$$

where as usual $0! = 1$ and $\delta^0(x) = x$ for all $x \in K$.

PROPOSITION 3. (i) $\theta_{\delta,n}$ *is an automorphism of* $K[\varepsilon_n]$ *as a* K_0*-algebra.*

(ii) $t_{\delta,n} = \theta_{\delta,n} \circ \iota$, *where* $\iota\colon k \to K$ *is the identity embedding*; *in particular*, $t_{\delta,n}$ *is an injective homomorphism of* k_0*-algebras.*

(iii) *For a nonzero derivation* $\delta\colon k \to K$, *the image of* $t_{\delta,n}$ *is dense in* $K[\varepsilon_n]$ *for the Zariski topology on* $K[\varepsilon_n]$ *as an* $(n+1)$*-dimensional affine space over* K.

Proof. In the first assertion, only the multiplicativity of $\theta_{\delta,n}$ requires verification for which we will use the "Leibnitz rule,"

$$\delta^j(xy) = \sum_{t=0}^{j}\binom{j}{t}\delta^t(x)\delta^{j-t}(y). \quad (3)$$

Let $a = \sum_{i=0}^n a_i\varepsilon_n^i$, $b = \sum_{i=0}^n b_i\varepsilon_n^i$. Then $ab = \sum_{i=0}^n c_i\varepsilon_n^i$, where $c_i = \sum_{s=0}^i a_s b_{i-s}$. On the other hand, $\theta_{\delta,n}(a)\theta_{\delta,n}(b) = \sum_{i=0}^n d_i\varepsilon_n^i$, where

$$d_i = \sum_{j=0}^{i}\left(\sum_{l=0}^{j}\frac{1}{l!}\delta^l(a_{j-l})\right)\left(\sum_{m=0}^{i-j}\frac{1}{m!}\delta^m(a_{i-j-m})\right).$$

A direct computation using (3) shows that

$$\begin{aligned} d_i &= \sum_{j=0}^{i} \sum_{s=0}^{i-j} \left(\sum_{t=0}^{j} \frac{1}{t!} \cdot \frac{1}{(j-t)!} \delta^t(a_s)\delta^{j-t}(a_{i-j-s}) \right) \\ &= \sum_{j=0}^{i} \sum_{s=0}^{i-j} \frac{1}{j!} \left(\sum_{t=0}^{j} \binom{j}{t} \delta^t(a_s)\delta^{j-t}(a_{i-j-s}) \right) \\ &= \sum_{j=0}^{i} \sum_{s=0}^{i-j} \frac{1}{j!} \delta^j(a_s a_{i-j-s}) = \sum_{j=0}^{i} \frac{1}{j!} \delta^j \left(\sum_{s=0}^{i-j} a_s a_{i-j-s} \right) \\ &= \sum_{j=0}^{i} \frac{1}{j!} \delta^j(c_{i-j}). \end{aligned}$$

It follows that $\theta_{\delta,n}(a)\theta_{\delta,n}(b) = \theta_{\delta,n}(c) = \theta_{\delta,n}(ab)$, as claimed.

Next, we claim that

$$\theta_{-\delta,n} \circ \theta_{\delta,n} = \mathrm{id}_{K[\varepsilon_n]}, \tag{4}$$

implying in particular that $\theta_{\delta,n}$ is an automorphism of $K[\varepsilon_n]$. Since $K[\varepsilon_n]$ is generated as a ring by K and ε_n, it suffices to check (4) separately on ε_n and K. But $\theta_{\delta,n}(\varepsilon_n) = \varepsilon_n = \theta_{-\delta,n}(\varepsilon_n)$, proving (4) for ε_n. Furthermore, for $a \in K$ we have

$$(\theta_{-\delta,n} \circ \theta_{\delta,n})(a) = \theta_{-\delta,n}\left(\sum_{i=0}^{n} \frac{1}{i!} \delta^i(a)\varepsilon_n^i \right) = \sum_{i=0}^{n} e_i \varepsilon_n^i,$$

where

$$e_i = \sum_{j=0}^{i} \frac{1}{j!}(-\delta)^j \left(\frac{1}{(i-j)!} \delta^{i-j}(a) \right) = \frac{1}{i!}(-\delta+\delta)^i(a).$$

So, $e_0 = a$ and $e_i = 0$ for $i > 0$, implying (4) on K.

Assertion (ii) is immediately obtained by comparing (1) and (2).

Since algebraic subgroups of $K[\varepsilon_n]$ coincide with (vector K-) subspaces, the fact that Im $t_{\delta,n}$ is not Zariski dense in $K[\varepsilon_n]$ would mean that there exists a nontrivial relation

$$\sum_{i=0}^{n} a_i \delta^i(x) = 0 \tag{5}$$

for all $x \in k$, where $a_i \in K$.

First, suppose that there exists $c \in k$ such that $\delta(c) = 1$. Then $\delta^i(c^i) = i!$ for any $i \geq 0$, hence $\delta^j(c^i) = 0$ for all $j > i$. Let $i_0 \in \{0, \dots, n\}$ be the smallest index with the property $a_{i_0} \neq 0$. Using (5) for $x = c^{i_0}$, we obtain

$$0 = a_{i_0} \delta^{i_0}(c^{i_0}) = (i_0)! a_{i_0},$$

implying $a_{i_0} = 0$, a contradiction.

In the general case, pick $c \in k$ such that $\alpha := \delta(c) \neq 0$ and introduce the derivation $\omega(x) = \alpha^{-1}\delta(x)$. An easy inductive argument shows that for any $l > 1$ there exists a relation of the form

$$\omega^l(x) = \sum_{i=1}^{l} \beta_i \delta^i(x) \qquad \text{for all } x \in k$$

with $\beta_i \in K$. It follows that there exists a linear transformation $\sigma\colon K[\varepsilon_n] \to K[\varepsilon_n]$ such that $t_{\omega, n} = \sigma \circ t_{\delta, n}$. Since $\omega(c) = 1$, our previous argument shows that $\operatorname{Im} t_{\omega, n}$ is Zariski dense in $K[\varepsilon_n]$, and the Zariski density of $\operatorname{Im} t_{\delta, n}$ follows.

In the sequel we will refer to $t_{\delta, n}$ as the *nth Taylor homomorphism* associated with the derivation δ (we note that for $n > 1$ the homomorphism $t_{\delta, n}$ may depend not only on the derivation $\delta\colon k \to K$ but also on the choice of an extension of δ to K).

For the construction of the group $\mathscr{G}_n$ we will describe the operation of restriction of scalars $R_{\ell[\varepsilon_n]/\ell}$, where ℓ is an arbitrary field, in explicit terms. For the m-dimensional affine space $\mathbb{A}^m$ over the universal domain Ω, we let $\mathscr{A}_n^m$ denote the m-dimensional affine space over the algebra $\Omega[\varepsilon_n]$ considered as the $(n+1)m$-dimensional affine space over Ω. Given a point $a \in \mathbb{A}^m$, for any integer l between 1 and $n+1$ we let

$$\mathscr{A}_n^m(a, l) = \{x = (x_1, \dots, x_m) \in \mathscr{A}_n^m \mid x_i \equiv a_i (\operatorname{mod} \varepsilon_n^l \Omega[\varepsilon_n]) \\ \text{for all } i = 1, \dots, m\}.$$

Given a Zariski closed subvariety $V \subset \mathbb{A}^m$, we let $\mathscr{V}_n$ denote the closed subvariety of $\mathscr{A}_n^m$ consisting of all zeros in $(\Omega[\varepsilon_n])^m$ of the ideal defining V. (Of course, $\mathscr{V}_n$ in none other than $R_{\Omega[\varepsilon_n]/\Omega}(V)$; besides, if V is defined over a subfield $\ell \subset \Omega$, then so is $\mathscr{V}_n$, and $\mathscr{V}_n = R_{\ell[\varepsilon_n]/\ell}(V)$.) For a point $a \in V$ and a number l between 1 and $n+1$, we let $\mathscr{V}_n(a, l) = \mathscr{V} \cap \mathscr{A}_n^m(a, l)$. For convenience of further reference we will record in the following lemma some elementary properties of these constructions.

LEMMA 8. (i) *Let $f\colon \mathbb{A}^{m_1} \to \mathbb{A}^{m_2}$ be a rational map. Then*

(1) *for any $n \geq 1$, f induces a rational map $f_n\colon \mathscr{A}_n^{m_1} \to \mathscr{A}_n^{m_2}$, and moreover, if f is defined over a subfield $\ell \subset \Omega$, then so is f_n;*

(2) *if f is defined at $a \in \mathbb{A}^{m_1}$, then f_n is defined on $\mathscr{A}^{m_1}(a, 1)$;*

(3) *if* f *is defined at* a, *then for any* $l, 1 \leq l \leq n$, *one has* $f_n(\mathscr{A}^{m_1}(a,l)) \subset \mathscr{A}^{m_2}(f(a),l)$; *more precisely, for* $x = a + a_l\varepsilon_n^l + \cdots + a_n\varepsilon_n^n \in \mathscr{A}^{m_1}(a,l)$, *where* $a, a_i \in \mathbb{A}^m$, *one has*

$$f_n(x) = f(a) + d_a f(a_l)\varepsilon_n^l + \cdots,$$

where $d_a f\colon \mathbb{A}^{m_1} \to \mathbb{A}^{m_2}$ *is the differential of* f *at* a *and* $\cdots$ *stands for the terms involving higher powers of* ε_n.

(ii) *Let* $V \subset \mathbb{A}^m$ *be a closed subvariety. If* $\tilde{a} = a + a_l\varepsilon_n^l + \cdots \in \mathscr{V}_n$, *where* $a, a_i \in \mathbb{A}^m$, *then* $a \in V$ *and* $a_l \in T_a(V)$ (*the tangent space*).

Proof. Let $g \in \Omega[t_1, \ldots, t_m]$ be a polynomial. We first observe that given a point $\tilde{a} = a + a_1\varepsilon_n + \cdots + a_n\varepsilon_n^n \in \mathscr{A}_n^m$, where $a, a_i \in \mathbb{A}^m$, there exist polynomials $G_1, \ldots, G_n$ in $(n+1)m$ variables such that

$$g(\tilde{a}) = g(a) + G_1(\tilde{a})\varepsilon_n + \cdots + G_n(\tilde{a})\varepsilon_n^n,$$

so g extends to a regular map $\mathscr{A}_n^m \to \mathscr{A}_n^1$. Furthermore, the equation

$$\left(\frac{1}{g}\right)(\tilde{a}) = g(a)^{-1}\left[1 - \frac{G(\tilde{a})}{g(a)} + \left(\frac{G(\tilde{a})}{g(a)}\right)^2 - \cdots + (-1)^n\left(\frac{G(\tilde{a})}{g(a)}\right)^n\right], \quad (6)$$

where $G(\tilde{a}) = G_1(\tilde{a})\varepsilon_n + \cdots + G_n(\tilde{a})\varepsilon_n^n$, shows that $(1/g)$ defines a rational map $\mathscr{A}_n^m \to \mathscr{A}_n^1$. It follows that a rational map $f\colon \mathbb{A}^{m_1} \to \mathbb{A}^{m_2}$ extends to a rational map $f_n\colon \mathscr{A}_n^{m_1} \to \mathscr{A}_n^{m_2}$, defined over the same subfield $\ell \subset \Omega$ as f, and moreover the explicit expression (6) for the inverse shows that if f is defined at $a \in \mathbb{A}^{m_1}$, then f_n is defined at any point of $\mathscr{A}^{m_1}(a, 1)$.

Now, if $\tilde{a} \in \mathscr{A}^m(a,l)$, then for any polynomial $g \in \Omega[t_1, \ldots, t_m]$ one has $g(\tilde{a}) \equiv g(a) \pmod{\varepsilon_n^l\Omega[\varepsilon_n]}$. It follows that if g/h is a rational function ($g, h \in \Omega[t_1, \ldots, t_m]$) such that $h(a) \neq 0$, then for any point $\tilde{a} \in \mathscr{A}_n^m$ one has

$$(g/h)(\tilde{a}) \equiv (g/h)(a) (\mathrm{mod}\ \varepsilon_n^l\Omega[\varepsilon_n]).$$

For a rational map $f\colon \mathbb{A}^{m_1} \to \mathbb{A}^{m_2}$ this implies the inclusion $f_n(A^{m_1}(a,l)) \subset \mathscr{A}^{m_2}(f(a),l)$. More explicitly, by Taylor's formula for the polynomials one has

$$g(a + a_l\varepsilon_n^l + \cdots) = g(a) + ((\mathrm{grad}_a\, g)\cdot a_l)\varepsilon_n^l + \cdots$$

and

$$h(a + a_l\varepsilon_n^l + \cdots) = h(a) + ((\mathrm{grad}_a\, h)\cdot a_l)\varepsilon_n^l + \cdots,$$

where $\mathrm{grad}_a = ((\partial/\partial t_1)(a), \ldots, (\partial/\partial t_m)(a))$, and $\cdot$ is the standard inner product on $\mathbb{A}^m$. It follows that

$$\begin{aligned}
&(g/h)(a+a_l\varepsilon_n^l+\cdots)\\
&\quad=(g(a)+((\mathrm{grad}_a g)\cdot a_l)\varepsilon_n^l+\cdots)h(a)^{-1}\left(1-\left(\frac{(\mathrm{grad}_a h)}{h(a)}\cdot a_l\right)\varepsilon_n^l+\cdots\right)\\
&\quad=(g/h)(a)+\left(\left[\frac{h(a)(\mathrm{grad}_a g)-g(a)(\mathrm{grad}_a h)}{h(a)}\right]\cdot a_l\right)\varepsilon_n^l+\cdots\\
&\quad=(g/h)(a)+(\mathrm{grad}_a(g/h)\cdot a_l)\varepsilon_n^l+\cdots.
\end{aligned}$$

This computation implies that, given a rational map $f\colon \mathbb{A}^{m_1} \to \mathbb{A}^{m_2}$ defined at a point $a \in \mathbb{A}^{m_1}$, one has

$$f_n(a + a_l\varepsilon_n^l + \cdots) = f(a) + d_af(a_l)\varepsilon_n^l + \cdots.$$

For (ii), we observe that for any polynomial g that vanishes on V and any point $a + a_l\varepsilon_n^l + \cdots \in \mathscr{V}_n(a, l)$ one has

$$0 = g(a + a_l\varepsilon_n^l + \cdots) = g(a) + ((\mathrm{grad}_a\, g)\cdot a_l)\varepsilon_n^l + \cdots,$$

implying that a_l is orthogonal to $\mathrm{grad}_a\, g$. This being true for any polynomial g that vanishes on V, we obtain that $a_l \in T_a(V)$.

Construction of the Group $\mathscr{G}_n$. Let G be an algebraic group defined over a subfield $\ell \subset \Omega$. We fix a matrix realization $G \subset GL_d$ for which G is Zariski–closed also in the matrix algebra $M_d \simeq \mathbb{A}^{d^2}$. Then the above construction yields a closed ℓ-subvariety $\mathscr{G}_n \subset \mathscr{M}_{d,n} = M_d(\Omega[\varepsilon_n])$ which turns out to be a group. In fact, this group is none other than $R_{\ell[\varepsilon n]/\ell}(G)$; i.e., for any ℓ-algebra A one has $\mathscr{G}_n(A) = G(A[\varepsilon_n])$. For any l between 1 and $n+1$, $\mathscr{G}_n(l) := \mathscr{G}(E_d, l)$ is a closed normal subgroup of $\mathscr{G}_n$ ("congruence subgroup of level l"); observe that $\mathscr{G}_n(n+1) = \{E_d\}$.

Assume from now on that G is connected. Then it is known (cf. [Bo, Theorem 18.2(ii)]) that G is ℓ-unirational; i.e., there exists a ℓ-defined dominant rational map $f\colon \mathbb{A}^m \to G$.

PROPOSITION 4. (1) *$\mathscr{G}_n/\mathscr{G}_n(1) \simeq G$, and $\mathscr{G}_n(l)/\mathscr{G}_n(l+1) \simeq L(G)$, the Lie algebra of G, for any l between* 1 *and n.*

(2) *The group $\mathscr{G}_n$ is connected of dimension $(n+1)\dim G$.*

(3) *The map $f_n\colon \mathscr{A}_n^m \to \mathscr{G}_n$ is dominant.*

(4) *If G is reductive then $\mathscr{G}_n(1)$ is the unipotent radical of $\mathscr{G}_n$.*

(5) *If G is semi-simple, then $\mathscr{G}_n(1)$ has nilpotency class n.*

Proof. (1) For $l = 1$, the group $\mathcal{G}_n(1)$ coincides with the kernel of the homomorphism $\mathcal{G}_n \to G$ induced by the ring homomorphism $\Omega[\varepsilon_n] \to \Omega$ sending ε_n to zero. Since the latter admits a cross-section $\Omega \to \Omega[\varepsilon_n]$, then so does the homomorphism $\mathcal{G}_n \to G$, proving that $\mathcal{G}_n/\mathcal{G}_n(1) \simeq G$.

Now, let $l \geq 1$. If $E_d + A\varepsilon_n^l + \cdots \in \mathcal{G}_n(l)$, then $A \in L(G)$ (cf. Lemma 8(ii)), and the correspondence

$$E_d + A\varepsilon_n^l + \cdots \mapsto A$$

defines a group homomorphism κ: $\mathcal{G}_n(l) \to L(G)$ whose kernel is precisely $\mathcal{G}_n(l+1)$. So, it remains to establish the surjectivity of κ. For this purpose we will use the dominant ℓ-defined map f: $\mathbb{A}^m \to G$. By modifying f by a (left) translation we can ensure that there exists a point $a \in \mathbb{A}^m(\ell)$ such that $f(a) = E_d$ and the differential $d_a f$ is surjective (because *char* $\ell = 0$; cf. [Bo, Sect. AG, Theorem 17.3]). According to Lemma 8(i), part (3), for $a + X\varepsilon_n^l \in \mathcal{A}_n^m(a, l)$, one has

$$f_n(a + X\varepsilon_n^l) = E_d + (d_a f)(X)\varepsilon_n^l + \cdots,$$

so the surjectivity of $d_a f$ implies that of κ, competing the proof of Assertion 1.

Assertion 2 immediately follows from Assertion 1. For Assertion 3, we observe that, as follows from our proof of Assertion 1, for any l between 1 and n the composition

$$T_a(\mathcal{A}_n^m(a, l)) \xrightarrow{d_a f_n} T_{E_d}(\mathcal{G}_n(l)) \longrightarrow T_{E_d}(\mathcal{G}_n(l)/\mathcal{G}_n(l+1))$$

is surjective. Since $\mathcal{G}_n(n+1) = \{E_d\}$, we conclude that $T_a(\mathcal{A}_n^m(a, 1)) \xrightarrow{d_a f_n} T_{E_d}(\mathcal{G}_n(1))$ is surjective. In conjunction with the surjectivity of $d_a f$, this implies the surjectivity of $d_a f_n$: $T_a(\mathcal{A}_n^m) \to T_{E_d}(\mathcal{G}_n)$, and the dominancy of f_n follows (cf. [Bo, Sect. AG, Theorem 17.3]).

Since all sections $\mathcal{G}_n(l)/\mathcal{G}_n(l+1)$, $l \geq 1$, are unipotent, the group $\mathcal{G}_n(1)$ is a connected unipotent normal subgroup in $\mathcal{G}_n$. So, if $G \simeq \mathcal{G}_n/\mathcal{G}_n(1)$ is reductive, $\mathcal{G}_n(1)$ is the unipotent radical of $\mathcal{G}_n$. Finally, suppose G is semisimple. We will need the following well-known and easily verifiable commutator relation: given $g = E_d + A\varepsilon_n + \cdots \in \mathcal{G}_n(1)$, $h = E_d + B\varepsilon_n^l + \cdots \in \mathcal{G}_n(l)$, then

$$ghg^{-1}h^{-1} = E_d + (AB - BA)\varepsilon_n^{l+1} + \cdots.$$

Since $L(G) = [L(G), L(G)]$, it easily follows that $[\mathcal{G}_n(1), \mathcal{G}_n(l)] = \mathcal{G}_n(l+1)$ for any $l \geq 1$, implying that the nilpotency class of $\mathcal{G}_n(1)$ is n.

Proof of Theorem 6. We return to the notations introduced in the statement of Theorem 6. Fix an $n \geq 1$ and consider the k_0-group $\mathcal{G}_n$ constructed above. Then all properties of $\mathcal{G}_n$ listed in the theorem have already been established in Proposition 4. So, it remains to construct a homomorphism $G(k) \to \mathcal{G}_n(K)$ with a Zariski dense image. For this we will use the nth Taylor homomorphism $t_{\delta,n}$: $k \to K[\varepsilon_n]$ constructed in Proposition 3. Since G is k_0-defined, $t_{\delta,n}$, being a homomorphism of k_0-algebras, induces a group homomorphism β: $G(k) \to G(K[\varepsilon_n]) = \mathcal{G}_n(K)$, and we only need to establish that β has Zariski dense image. Let f: $\mathbb{A}^m \to G$ be a k_0-defined dominant rational map as above. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{A}^m(k) & \xrightarrow{\alpha} & \mathbb{A}^m(K[\varepsilon]) = \mathcal{A}_n^m(K) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_n} \\ G(k) & \xrightarrow{\beta} & G(K[\varepsilon_n]) = \mathcal{G}_n(K) \end{array}$$

in which α, just like β, is induced by $t_{\delta,n}$. Proposition 3(iii) implies the density of the image of α. On the other hand, according to Proposition 4(iii), f_n is dominant, and the density of the image of β follows.

We conclude this section with an observation which generalizes for our setup the observation made by Tits [T, 2.4] that the group of motions of three-dimensional euclidean space has (a lot of) noncontinuous automorphisms. Let K be a field having a nonzero derivation δ: $K \to K$, K_0 the field of constants of δ, and G a connected algebraic K_0-group. Fix an $n \geq 1$. Then the automorphism θ_δ of $K[\varepsilon]$ (where $\varepsilon^{n+1} = 0$) constructed in Proposition 3 induces a group automorphism Θ_δ of $\mathcal{G}_n = G(K[\varepsilon])$. Moreover, Θ_δ maps the subgroup $G(K) \subset \mathcal{G}_n(K)$, corresponding to the identity embedding $K \hookrightarrow K[\varepsilon]$, to the subgroup $H \subset \mathcal{G}_n(K)$ obtained by using the Taylor homomorphism $t_{\delta,n}$: $K \to K[\varepsilon]$. Thus, Θ_δ takes a subgroup whose Zariski closure has dimension $\dim G$ to a subgroup with Zariski closure having dimension $(n+1) \cdot \dim G$, so Θ_δ is strongly "noncontinuous."

Afterword. In Part II of this paper we extend the notion of Taylor homomorphisms to the case of "several variables." More precisely, given l derivations δ_i: $k \to K$, $i = 1, \ldots, l$, we will construct for any $n \geq 1$ a ring homomorphism $t_{n,\delta_1,\ldots,\delta_l}$: $k \to K[\varepsilon_1, \ldots, \varepsilon_l]$ where all ε_i's commute and satisfy $\varepsilon_i^{n+1} = 0$. If the derivations δ_i are linearly independent over K, the image of $t_{n,\delta_1,\ldots,\delta_l}$ is going to be Zariski dense in $K[\varepsilon_1, \ldots, \varepsilon_l]$, regarded as the $(n+1)^l$-dimensional affine space over K. Then the proof of Theorem 6 extends without any changes to show that, for any connected algebraic group G defined over the field of constants k_0 of all δ_i's, the homomorphism $t_{n,\delta_1,\ldots,\delta_l}$ induces an abstract homomorphism $G(k) \to G(K[\varepsilon_1, \ldots, \varepsilon_l])$ with Zariski dense image. Furthermore, we will show that *any* ring homomorphism $k \to K[\varepsilon_1, \ldots, \varepsilon_l]$ can be obtained from a suitable Taylor homomorphism by a sequence of so-called twists and then will derive from Theorem 3 by induction on l that, for G a simple simply connected

Chevalley group, *any* abstract homomorphism $G(k) \to G(K[\varepsilon_1, \ldots, \varepsilon_l])$ of the form (1), Section 1, corresponds to a certain ring homomorphism $k \to K[\varepsilon_1, \ldots, \varepsilon_l]$. Finally, we will show that any abstract homomorphism $G(k) \to \mathscr{G}(K)$ of the form (1), Section 1, factors through a homomorphism $G(k) \to G(A)$, where A is a K-algebra having structure similar to that of $K[\varepsilon_1, \ldots, \varepsilon_l]$, which will complete the explicit description of abstract homomorphisms of Chevalley groups in characteristic zero.

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